Geometry of deformations of branes in warped backgrounds

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Abstract

The 'braneworld' (described by the usual worldvolume action) is a D dimensional timelike surface embedded in a N dimensional ($N > D$) warped, nonfactorisable spacetime. We first address the conditions on the warp factor required to have an extremal flat brane in a five dimensional background. Subsequently, we deal with normal deformations of such extremal branes. The ensuing Jacobi equations are analysed to obtain the stability condition. It turns out that to have a stable brane, the warp factor should have a minimum at the location of the brane in the given background spacetime. To illustrate our results we explicitly check the extremality and stability criteria for a few known co-dimension one braneworld models. Generalisations of the above formalism for the cases of (i) curved branes (ii) asymmetrical warping and (iii) higher co-dimension braneworlds are then presented alongwith some typical examples for each. Finally, we summarize our results and provide perspectives for future work along these lines.

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I. INTRODUCTION

Models with extra dimensions have been around for quite some time now. Beginning with Kaluza–Klein [1], unified theories and followed in the later parts of the twentieth century by string theory [2], a large variety of models with extra dimensions have been proposed. However, till now, we are yet to find any experimental verification of the existence of such dimensions. Among recent developments, we have had the so–called braneworld models, which are, primarily, of two types–flat spacetime models (the large extra dimension scenario) [3] and the warped models [4]. In the latter, we find that the four dimensional submanifold (3-brane) has a dependence on the extra dimension through the so–called warp factor, which, in turn makes the background line element nonfactorisable.

All of the currently popular braneworld scenarios assume that we live on a 3–brane in a background spacetime. This viewpoint leads us to the fact that our world is an *embedded* hypersurface (the 3–brane) in a five dimensional background (called the 'bulk'). One may also view the 3–brane as a domain wall or a defect in spacetime. An extensive literature exists on such domain walls in supergravity theories [5].

The action for such an embedded hypersurface would therefore be the usual 'worldvolume' action (dimensionally extending the standard Nambu–Goto area action for strings) [2]. It is thus relevant to ask whether such hypersurfaces are extremal (vanishing of the trace of the extrinsic curvature, $K = 0$) and if so, whether they are stable against small normal deformations. The aim of this article is to address these issues in the context of the warped braneworld models [4].

We first write down the condition for extremality mentioned above for arbitrary embedded surfaces in a warped background geometry. Then, for different types of warping, we investigate whether this extremality condition holds good. Subsequently, we move on to the so–called Jacobi equations (obtained by constructing a second variation of the worldvolume action) which describe the perturbative normal deformations of the embedded surface. The solutions for these equations address the question of stability through the nature of the modes–oscillatory modes indicate stability whereas growing modes confirm the existence of an instability. We check these consequences for the various warped braneworld models with one extra dimension that exist in the literature. Having dealt with the flat branes (induced metric being scaled Minkowski) we move on to discuss two types of curved

branes–the spatially flat cosmological braneworld and the spherically symmetric, static one. The extremality and stability criteria for such curved branes are obtained for some special cases. Finally, we analyse various possible extensions (eg. asymmetrically warped branes and higher co-dimension branes) and summarize our results in brief.

II. EXTREMALITY AND PERTURBATIONS ON WORLDSHEET

We consider a D-dimensional hypersurface (worldsheet) embedded in a N-dimensional background spacetime. The coordinates of the background are $X^\mu = X^\mu(\xi^a)$ where ξ^a are the coordinates on the worldsheet. Here $a \equiv 1...D$ are the worldsheet indices, $\mu \equiv 1...N$ the background indices and $i \equiv 1...(N-D)$ are the normal indices.

With the aid of the orthonormal spacetime basis (E^μ_a, n^μ_i) consisting of D tangents and $(N-D)$ normals satisfying the orthonormality conditions

$$g_{\mu\nu}E^\mu_a E^\nu_b = \eta_{ab} \tag{2.1}$$

$$g_{\mu\nu}n^\mu_i n^\nu_j = \delta_{ij} \tag{2.2}$$

$$g_{\mu\nu}E^\mu_a n^\nu_i = 0 \tag{2.3}$$

the extrinsic curvature tensor components of the worldsheet is defined as

$$K^i_{ab} = -g_{\mu\nu}E^\alpha_a(\nabla_\alpha E^\mu_b)n^{\nu i} \tag{2.4}$$

where $g_{\mu\nu}$ is the background metric and ∇_μ are the covariant derivatives with respect to the background coordinates. The usual Nambu-Goto area action for the worldsheet is given by

$$S = -\lambda \int d^D\xi\sqrt{-\gamma} \tag{2.5}$$

where $\gamma_{ab} = g_{\mu\nu}X^\mu_{,a}X^\nu_{,b}$ are the induced metric coefficients on the worldsheet. The equation of motion (EOM) for the worldsheet in the higher dimensional background is obtained by extremising S subject to the deformation [6, 7]

$$X^\mu(\xi^a) \to X^\mu(\xi^a) + \delta X^\mu(\xi^a) \tag{2.6}$$

For a minimal surface, the first variation of the action gives the EOM comprising of $(N-D)$ equations involving the trace of the extrinsic curvature [7] :

$$\gamma^{ab} K^i_{ab} = 0 \tag{2.7}$$

It is worth mentioning here that $(N-D)$ is the number referred to as the co-dimension of the embedded worldsheet.

Unlike the first variation equation, the procedure of finding out the second variation equation is a bit tedious. Still, for the sake of completeness, here we provide the major steps of calculation. Since only the motion transverse to the worldsheet is relevant (tangential perturbations can be gauged away via reparametrisation invariance) one can write [9, 10]

$$\delta X^\mu = \Phi^i n^{\mu i} \tag{2.8}$$

in terms of the $(N-D)$ scalars Φ^i. The second variation of the action is given by

$$\delta^2 S = \lambda \int d^D\xi \ \left[g_{\mu\nu} \delta^\mu \mathcal{D}_\delta \mathcal{D}_a (\sqrt{-\gamma}\gamma^{ab} E^\nu_b) \right] \tag{2.9}$$

where $\delta = \delta X^\mu \partial_\mu$ is the spacetime vector field and the derivatives $\mathcal{D}_a, \mathcal{D}_\delta$ are defined as $\mathcal{D}_a = X^\mu_{,a} \nabla_\mu$ and $\mathcal{D}_\delta = \delta X^\mu \nabla_\mu$ respectively. We next project the term inside square-bracket of Eq (2.9) onto the unit normal $n^{\nu i}$, that results in the following equation

$$g_{\mu\nu} n^{\mu i} \mathcal{D}_\delta \mathcal{D}_a (\sqrt{-\gamma}\gamma^{ab} E^\nu_b) = 0 \tag{2.10}$$

and utilise the spacetime Ricci identity to obtain

$$\mathcal{D}_\delta \mathcal{D}_a (\sqrt{-\gamma}\gamma^{ab} E^\nu_b) = \mathcal{D}_a \mathcal{D}_\delta (\sqrt{-\gamma}\gamma^{ab} E^\nu_b) + R E^\mu_a \delta_\mu (\sqrt{-\gamma}\gamma^{ab} E^\nu_b) \tag{2.11}$$

Further, with the help of the projection tensor $h^{\mu\nu} = g^{\mu\nu} - n^{\mu i} n^{\nu i}$, we write

$$\gamma^{ab} g_{\mu\nu} \delta^\mu (R E^\alpha_a \delta_\alpha) E^\nu_b = \left[R_{\mu\nu} n^{\mu i} n^{\nu j} - R_{\mu\alpha\nu\beta} n^{\mu i} n^{\alpha k} n^{\nu j} n^{\beta k} \right] \Phi^i \Phi^j \tag{2.12}$$

where the Riemann and Ricci curvature tensor components and the Ricci scalar are those of the background metric. Subsequently, we substitute the results of Eq (2.11) and (2.12) into Eq (2.10), decompose the first term into three parts and analyse each part by introducing the surface torsion $T^{ij}_a = g_{\mu\nu}(\mathcal{D}_a n^{\mu i}) n^{\nu j} = -T^{ji}_a$. As a result, the second variation (written as an action now) turns out to be

$$S = \frac{1}{2}\delta^2 S = \frac{1}{2} \int d^D\xi \sqrt{-\gamma} \left[\Phi^k \Box \Phi^i - \Phi^k (M^2)^i_j \Phi^j \right] \tag{2.13}$$

where, $\Box$ is the worldsheet d'Alembertian and we have absorbed all the torsion terms by re-defining the worldsheet derivative as $\tilde{\nabla}_a^{ij} = \nabla_a \delta^{ij} - T_a^{ij}$. Also, we have defined the effective mass matrix $(M^2)^{ij}$ to be

$$(M^2)^{ij} = R_{\mu\alpha\nu\beta} n^{\mu i} n^{\alpha k} n^{\nu j} n^{\beta k} - R_{\mu\nu} n^{\mu i} n^{\nu j} - K^i_{ab} K^{abj} \tag{2.14}$$

Finally, the variation of the action with respect to Φ^k leads to the following equation [8]

$$\Box \Phi^i - (M^2)^i_j \Phi^j = 0 \tag{2.15}$$

What turns out from the above equation is that for a Minkowski brane, a negative eigenvalue of $(M^2)^{ij}$ will lead to an instability.

Furthermore, for a $(N-1)$ dimensional hypersurface (co-dimension 1) with a single normal vector, both the torsion and the total projected Riemann tensor onto the normal vanish, and Eq (2.15) reduces to [8]

$$\Box \Phi + (R_{\mu\nu} n^\mu n^\nu + K_{ab} K^{ab}) \Phi = 0 \tag{2.16}$$

Equations (2.7) and (2.16) are the two guiding principles in discussing the extremality and stability of branes. In a nutshell, in order that the worldsheet be extremal and stable, the extrinsic curvature tensor has to be traceless and the effective mass matrix needs to have positive eigenvalues.

III. SCHEMATICS OF WARPED SPACETIMES

Let us now switch over to a brief discussion of the geometry of warped extra dimensions. We consider the by-now well-known Randall-Sundrum type [4] braneworlds where our 4D observable universe (called a brane) is a hypersurface embedded in a 5D geometry (with a bulk negative cosmological constant) where for the RS1 two brane model (branes at 0 and πr_c in the extra dimension) the extra dimension constitutes an S^1/Z_2 orbifold and for RS2 the extra dimension is infinite. The various parameters in RS1 (brane tensions, bulk cosmological constant, curvature scale) are fine-tuned in such a way that the effective brane cosmological constant Λ_4 turns out to zero. The tensions on the so-called visible (at $\sigma = \pi r_c$) and hidden (at $\sigma = 0$) branes are of equal magnitude but of negative and positive signs respectively. Subsequently, a number of thin and thick brane models [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]

have been proposed by introducing different forms of bulk scalars and brane tensions, that can play the role of a bulk cosmological constant. Proposals with Z_2-asymmetry also exist [21].

In this so-called warped geometric setup where the curvature is there in the bulk but the brane is flat, the 5-dimensional background metric can be expressed as

$$dS_5^2 = e^{2f(\sigma)}\eta_{\mu\nu}dX^\mu dX^\nu + d\sigma^2 \tag{3.1}$$

where $f(\sigma)$ is the "warp factor" that incorporates the effect of the extra-dimension σ on the brane and renders the bulk geometry nonfactorisable.

The induced metric on the brane is given by

$$dS_I^2 = e^{2f(\sigma_0)}\eta_{ab}d\xi^a d\xi^b \tag{3.2}$$

where $\sigma = \sigma_0$ is the location of the brane. The form of the metric shows that at $\sigma = \sigma_0$ the coordinates are re-scaled so that we get back the usual 4D flat geometry.

The basic questions we are now going to address are : what are the extremality and stability conditions in this warped geometry and which among the known braneworld solutions available in the literature are extremal and stable?

IV. EXTREMALITY AND STABILITY CONDITIONS OF BRANES

We denote $X^\mu \equiv (t, x, y, z, \sigma)$ for background coordinates and $\xi^a \equiv (\tau, x_1, y_1, z_1)$ for brane coordinates. With the background metric given by Eq (3.1) and the induced metric of Eq (3.2) obtained by the embedding at $\sigma = \sigma_0$, the non-zero components of the normalised tangent vectors turn out to be

$$E^t_\tau = E^x_{x_1} = E^y_{y_1} = E^z_{z_1} = e^{-f(\sigma)} \tag{4.1}$$

whereas the single normal vector is chosen as

$$n^\mu = (0, 0, 0, 0, 1) \tag{4.2}$$

With these expressions for tangent and normal vectors and keeping in mind that all the covariant derivatives are taken with respect to the background coordinates, the extrinsic curvature tensor turns out to be diagonal with the components

$$-K_{\tau\tau} = K_{x_1x_1} = K_{y_1y_1} = K_{z_1z_1} = f'(\sigma) \tag{4.3}$$

where a prime denotes derivative with respect to σ. Further, for co-dimension 1, the components of the Ricci tensor need to be evaluated. However, the only relevant component that has a non-trivial effect on the perturbation equation is

$$R_{\sigma\sigma} = -4\left[f^{''}(\sigma) + f^{'2}(\sigma)\right] \tag{4.4}$$

The 1st variation equation (2.7) requires a straightforward trace calculation of the extrinsic curvature, which gives

$$f^{'}(\sigma_0) = 0 \tag{4.5}$$

All it tells is that for the brane to be extremal $f(\sigma)$ should have an extremum at the location of the brane.

Let us now proceed further to exploit the 2nd variation equation in order to find the nature of the extremum. With the expressions for the Ricci tensor and extrinsic curvature given above, Eq (2.16), for an extremal brane, reduces to

$$\left[\Box - 4f^{''}(\sigma_0)\right]\Phi = 0 \tag{4.6}$$

Hence the stability requirement, *i.e.* a positive eigenvalue of the effective mass matrix [8], in the present scenario, reduces to

$$f^{''}(\sigma_0) > 0 \tag{4.7}$$

Our demand can further be justified by the following argument. Following standard textbook stuff we expect a wavelike solution for Eq (4.6) of the form (in terms of the scaled coordinates $\xi_I^a = e^{f(\sigma_0)}\xi^a$)

$$\Phi = A_0 e^{i(\omega\tau_I - \overrightarrow{k}.\overrightarrow{\xi}_I)} \tag{4.8}$$

where A_0 is a constant. Since Φ is an arbitrary scalar, equations (4.6) and (4.8) together give the oscillation frequency ω as

$$\omega^2 = k^2 + 4f^{''}(\sigma_0) \tag{4.9}$$

Hence the sufficient condition for the frequency to be real, and as a result, the brane to be stable is

$$f^{''}(\sigma_0) > 0 \tag{4.10}$$

justifying our demand in Eq (4.7). The constraint on $f^{''}(\sigma_0)$ further reveals that the extremum is, in fact, a minimum. However it should be mentioned that in no way this is a mandatory condition. The frequency can be real even when $f^{''}(\sigma_0)$ is negative if the condition

$$k^2 > 4|f^{''}(\sigma_0)| \tag{4.11}$$

is satisfied. Hence a positive value of $f^{''}(\sigma_0)$ only guarantees brane stability.

Equations (4.5) and (4.7) together provide the stability and extremality conditions for branes in a warped background. The bottomline is that to have a stable braneworld solution the warp factor should have a minimum and the brane has to be located at the minimum.

Once the stability condition is fixed, one can now easily find out the effect of perturbations by explicitly writing down the perturbed coordinates

$$X^{'\mu} = X^{\mu} + \Phi n^{\mu} \tag{4.12}$$

In the present scenario, all but one coordinate remain unperturbed. The lone perturbed coordinate is the extra dimension

$$\sigma^{'} = \sigma_0 + Re\ [A_0 e^{i(\omega\tau_I - \overrightarrow{k}.\overrightarrow{\xi}_I)}] \tag{4.13}$$

Hence a stable flat brane in a warped background oscillates harmonically about the stable location $\sigma = \sigma_0$ with a frequency $\omega = \sqrt{k^2 + 4f''(\sigma_0)}$.

A. Checking the stability of some co-dimension one models

The following table discusses the extremality and stability of a few co-dimension one braneworld solutions available in the literature [4, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

$f(\sigma)$	$f^{'}(\sigma_0)$	$f^{''}(\sigma_0)$	inference	extremal brane location	brane location in models	ω^2
$-k\|\sigma\|$ [4]	$\neq 0$	< 0	unstable	–	0	–
$\frac{1}{2}\ln[Ae^{-b\sigma} + Be^{b\sigma}]$ [11]	$= 0$	> 0	stable	$\frac{1}{2b}\ln\frac{A}{B}$	$\sigma_1, -\sigma_2$	$k^2 + 2b^2$
$\frac{c-\|\sigma\|}{L}$ [12]	$\neq 0$	< 0	unstable	–	0	–
$-\beta\ln\cosh^2(a\sigma)$ $+\frac{\beta}{2}\tanh^2(a\sigma)$[13]	$= 0$	< 0	stable if $k^2 > 12a^2\beta$	0	0	$k^2 - 12a^2\beta$
$\frac{1}{4}\ln(\frac{4}{3}\sigma + c) + d$ [14]	$\neq 0$	< 0	unstable	–	0	–
$c\ln\mathrm{sech}(b\sigma)$ [15, 16, 17]	$= 0$	< 0	stable if $k^2 > 4b^2c$	0	0	$k^2 - 4b^2c$
$c\ln\cosh(b\sigma)$ [18]	$= 0$	> 0	stable	0	0	$k^2 + 4b^2c$
$\frac{a}{k}e^{-k\|\sigma\|}$ [19]	$= 0$	$= 0$	extremal, unstable	$\mp\infty$	0	–

Some comments on the above table are in order. From the geometrical analysis, it is clear that very few braneworld solutions available in the literature are extremal as well as stable. For example, a study on the models in [4, 12] reveal that any linear solution for the warp factor is neither extremal nor stable, that re-establishes the instability of Randall-Sundrum model. Also, the stability requirement imposes extra constraints on the parameters involved in some of the braneworld models, *e.g,* [13, 15], satisfying the inequalities. Further, for models in [13, 15, 16, 17, 18], the brane locations in the respective models are their extremal and stable locations whereas for the model in [11], the stability criteria fix the brane location : $\sigma_1 = \frac{1}{2b}\ln(A/B)$ or $\sigma_2 = -\frac{1}{2b}\ln(A/B)$.

To conclude, in order to have a physically acceptable braneworld solution, one must check whether it is geometrically stable or not. Herein lies the importance of the present analysis.

V. GENERALISATIONS OF THE FORMALISM

So-far we discussed the stability of the usual symmetrical warping for flat branes with co-dimension one (hypersurfaces). Apart from this, several other braneworld models have been invoked to solve a few physical problems facing the earlier models and with the notion of extending the bulk-brane scenario where curvature on the brane or more than one extra

(spacelike) dimensions are involved. However, as discussed earlier, to have a valid braneworld solution one must check the stability of such models as well. With this intention we extend our formalism for the three different kinds of models, namely, (A) the braneworld models with curvature on the brane, (B) the asymmetrically warped spacetimes with co-dimension 1 and (C) the warped models with co-dimension 2.

A. Curved branes in warped backgrounds

Curved branes in warped backgrounds, reproducing the standard 4-dimensional geometry at the location of the brane, is of much physical interest from the cosmological and gravitational points of view. In this subsection, we intend to find out the extremality and stability conditions of these type of branes. More precisely, we explore two most important embedding geometries, one representing the spatially flat Friedman-Robertson-Walker metric on the brane, another a spherically symmetric brane representing the Schwarzschild spacetime.

1. Spatially flat FRW branes

The background metric for which the 4-dimensional cosmology with a spatially flat FRW brane is recovered, can be written as [22] (or in terms of transformed coordinates from [23])

$$dS_5^2 = e^{2f(\sigma)}[-dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)] + d\sigma^2 \tag{5.1}$$

whereas the induced metric representing the brane is given by

$$dS_I^2 = e^{2f(\sigma_0)}[-d\tau^2 + a^2(\tau)(dx_1^2 + dy_1^2 + dz_1^2)] \tag{5.2}$$

where, as usual, (t, x, y, z, σ) are the background coordinates and (τ, x_1, y_1, z_1) are the brane coordinates. Here $a(\tau)$ is the scale factor for spatially flat cosmological models. Later on we shall choose it to be de-Sitter in order to study a specific, solvable case explicitly.

The non-zero components of the normalised tangent vectors are

$$E^t_\tau = e^{-f(\sigma)} \quad ; \quad E^x_{x_1} = E^y_{y_1} = E^z_{z_1} = e^{-f(\sigma)}a^{-1}(t) \tag{5.3}$$

whereas the normal vector remains the same as for the flat brane. In this setup, the components of the extrinsic curvature tensor K_{ab} are $-K_{\tau\tau} = K_{x_1x_1} = K_{y_1y_1} = K_{z_1z_1} = f'(\sigma)$

and the relevant Ricci tensor component is $R_{\sigma\sigma} = -4[f''(\sigma) + f'^2(\sigma)]$. It turns out that they are identical to their corresponding expressions for a flat brane and although the other Ricci tensor components are different from the corresponding expressions for a flat brane, they do not affect the variation equations. As a result, the 1st variation equation involving the trace of K_{ab} leads to the extremality condition

$$f'(\sigma_0) = 0 \tag{5.4}$$

which is the same as that of the flat brane. And the 2nd variation equation gives

$$\left[\Box - 4f''(\sigma_0)\right]\Phi = 0 \tag{5.5}$$

Notice that the worldsheet d'Alembertian now includes curvature. Hence the stability requirement for a flat brane, *i.e,* a positive eigenvalue for the effective mass matrix, will not hold good in the present scenario. In order to find out the appropriate stability condition, we expand Eq (5.5) for the above worldsheet metric (in terms of the scaled coordinates $\xi_I^a = e^{f(\sigma_0)}\xi^a$), which reads

$$-\ddot{\Phi} + \frac{1}{a^2}\nabla^2\Phi - 3\frac{\dot{a}}{a}\dot{\Phi} - 4f''(\sigma_o)\Phi = 0 \tag{5.6}$$

where a dot denotes a derivative with respect to the scaled time and $\overrightarrow{\nabla}$ is the gradient in terms of the 3-space on the brane. The solution of the above equation is of the form $\Phi = A(\tau_I)e^{-i\overrightarrow{k}.\overrightarrow{\xi}_I}$, which, by transformation of the variable $A(\tau_I) = F(\tau_I)G(\tau_I)$, turns out to be

$$\Phi = a^{-3/2}F(\tau_I)\ e^{-i\overrightarrow{k}.\overrightarrow{\xi}_I} \tag{5.7}$$

with the function $F(\tau_I)$ satisfying the differential equation

$$\ddot{F} + \left[-\frac{3}{2}\frac{\ddot{a}}{a} - \frac{3}{4}(\frac{\dot{a}}{a})^2 + \frac{k^2}{a^2} + 4f''(\sigma_o)\right]F = 0 \tag{5.8}$$

Hence all one has to do is to solve for $F(\tau_I)$ with appropriate scale factor $a(\tau_I)$ and search for a decaying time-dependence of Φ, that will establish brane stability.

Let us now explicitly perform the stability analysis for a de-Sitter brane for which $a(\tau_I) = e^{H\tau_I}$ (H = Hubble constant). Here Eq (5.8) takes the form

$$\ddot{F} + \left[4f^{''}(\sigma_o) - \frac{9}{4}H^2 + k^2 e^{-2H\tau_I}\right] F = 0 \tag{5.9}$$

which has a Bessel function solution

$$F(\tau_I) = C_1 \; J_{-\nu}(\frac{k}{H}e^{-H\tau_I}) + C_2 \; J_{\nu}(\frac{k}{H}e^{-H\tau_I}) \tag{5.10}$$

where the order of the Bessel function is defined by $\nu = \frac{1}{H}\sqrt{\frac{9}{4}H^2 - 4f''(\sigma_0)}$ and C_1, C_2 are two arbitrary constants. With this, the expression for Φ turns out to be

$$\Phi = e^{-\frac{3}{2}H\tau_I}\left[C_1 \; J_{-\nu}(\frac{k}{H}e^{-H\tau_I}) + C_2 \; J_{\nu}(\frac{k}{H}e^{-H\tau_I})\right] e^{-i\overrightarrow{k}.\overrightarrow{\xi}_I} \tag{5.11}$$

The exponential term in the above expression decays with the worldsheet time whereas the Bessel functions are oscillatory. As a result, Φ turns out to be a decaying function of the worldsheet time, as desired by the stability criteria.

An extreme case is if ν happens to be an integer $(= n)$ *i.e,* when $4f^{''}(\sigma_0) = (9/4 - n^2)H^2$. In that case, Eq (5.9) has two linearly independent solutions, one a Bessel function of integer order n, another a Neumann function,

$$F(\tau_I) = C_1 \; J_n(\frac{k}{H}e^{-H\tau_I}) + C_2 \; Y_n(\frac{k}{H}e^{-H\tau_I}) \tag{5.12}$$

The Neumann function being divergent, the solution involving it is ruled out by the stability criteria.

We further notice that the order of the Bessel function has to be real. This gives the following stability condition for a de-Sitter brane

$$f^{''}(\sigma_0) \leq \frac{9}{16}H^2 \tag{5.13}$$

In brief, a de-Sitter brane is stable only if the warp factor at the location of the brane has certain upper bound determined by the parameter H.

The most popular de-Sitter brane embedded in warped background of the form of Eq (5.1) has the warp factor [22]

$$f(\sigma) = \ln\left(\sqrt{\Lambda}L\sinh\frac{c - |\sigma|}{L}\right) \tag{5.14}$$

Clearly, $f^{'}(\sigma) \neq 0$ for any value of $\sigma = \sigma_0$. Hence the brane has neither a minimum nor a maximum anywhere. So this is not an extremal brane. As a direct consequence, one cannot have any stable location for the brane too.

Even though, we have explicitly worked out the case for a de–Sitter, it should be clear that for other types of scale factors one can try to understand the extremality and stability of the corresponding cosmological brane using the formalism discussed above, though analytic solutions of the Jacobi equations may not always be necessarily available.

2. Spherically symmetric, static branes

For a spherically symmetric, static brane embedded in a warped background, the background metric can be expressed as [24]

$$dS_5^2 = e^{2f(\sigma)}[-e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega^2] + d\sigma^2 \tag{5.15}$$

with the induced metric of the form

$$dS_I^2 = e^{2f(\sigma_0)}[-e^{2\nu(r_1)}d\tau^2 + e^{2\lambda(r_1)}dr_1^2 + r_1^2 d\Omega_1^2] \tag{5.16}$$

Here the non-trivial normalised tangent vectors are

$$E_\tau^t = e^{-f(\sigma)-\nu(r)},\ E_{r_1}^r = e^{-f(\sigma)-\lambda(r)},\ E_{\theta_1}^\theta = \frac{e^{-f(\sigma)}}{r},\ E_{\phi_1}^\phi = \frac{e^{-f(\sigma)}}{r\sin\theta} \tag{5.17}$$

which give rise to the extrinsic curvature tensor components as $-K_{\tau\tau} = K_{r_1 r_1} = K_{\theta_1\theta_1} = K_{\phi_1\phi_1} = f'(\sigma)$ and the Ricci tensor component $R_{\sigma\sigma}$ is $-4[f^{''}(\sigma) + f^{'2}(\sigma)]$. As before, the perturbation equations are independent of the other Ricci tensor components. Consequently, the 1st and 2nd variation equations resemble Eq (5.4) and Eq (5.5) respectively. Here also the first equation provides us with the standard extremality condition but the second equation needs to be further studied in order to obtain the stability criteria. Considering radial perturbation $\Phi = \Phi(\tau_I,\ r_I)$, we explicitly write down the equation in terms of the scaled worldsheet coordinates $\xi_I^a = e^{f(\sigma_0)}\xi^a$.

$$-\frac{\partial^2\Phi}{\partial\tau_I^2} + e^{2(\nu-\lambda)}\frac{\partial^2\Phi}{\partial r_I^2} - e^{2(\nu-\lambda)}\left[\frac{2}{r_I} - \frac{d(\lambda-\nu)}{dr_I}\right]\frac{\partial\Phi}{\partial r_I} - 4f^{''}(\sigma_0)e^{2\nu}\Phi = 0 \tag{5.18}$$

Its solution is of the form $\Phi = e^{i\omega\tau_I} B(r_I)$ which, by re-writing the variable $B(r_I)$ as $B(r_I) = F(r_I)G(r_I)$, becomes

$$\Phi = \frac{e^{(\lambda-\nu)/2}}{r_I}\ F(r_I)\ e^{i\omega\tau_I} \tag{5.19}$$

where $F(r_I)$ now satisfies the equation

$$\ddot{F} + \left[\frac{1}{2}(\ddot{\lambda} - \ddot{\nu}) - \frac{1}{4}(\dot{\lambda} - \dot{\nu})^2 - \frac{1}{r}(\dot{\lambda} - \dot{\nu}) + \omega^2 e^{2(\lambda-\nu)} - 4f^{''}(\sigma_0)e^{2\lambda}\right] F = 0 \tag{5.20}$$

Here a dot denotes a derivative with respect to r_I. Once again the stability will be established if Φ is found to be a decaying function of the worldsheet radial distance r_I.

Specifically, the field outside a spherically symmetric gravitating body is given by the exterior Schwarzschild metric, for which the above equation reduces to

$$\ddot{F} + \left[\frac{\frac{M^2}{r_I^4} + \omega^2}{(1 - \frac{2M}{r_I})^2} - \frac{4f^{''}(\sigma)}{(1 - \frac{2M}{r_I})}\right] F = 0 \tag{5.21}$$

At a distance much larger than its Schwarzschild radius ($r_I \gg 2M$), that is relevant for most of the gravitating bodies since their Schwarzschild radii lie well within the actual radii, Eq (5.21) can be recast as

$$-\ddot{F} - \frac{4M[\omega^2 - 2f^{''}(\sigma_0)]}{r_I} F = [\omega^2 - 4f^{''}(\sigma_0)]F \tag{5.22}$$

that looks like the well-known Hydrogen atom problem with $V(r_I) = -\frac{4M}{r_I}[\omega^2 - 2f^{''}(\sigma_0)] < 0$. To have a stable brane, one has to search for the $l = 0$ bound state solutions, for which $E = \omega^2 - 4f^{''}(\sigma_0) < 0$. Hence in the lowest order [25]

$$F(r_I) = R_{10}(r_I) = Ne^{-kr_I}\ {}_1F_1(0,\ 2,\ 2kr_I) \tag{5.23}$$

where ${}_1F_1(a,\ b,\ z)$ is the confluent hypergeometric function. Consequently,

$$\Phi = N\frac{1}{r_I\sqrt{1 - \frac{2M}{r_I}}}\ e^{-kr_I}\ {}_1F_1(0,\ 2,\ 2kr_I)\ e^{i\omega\tau_I} \tag{5.24}$$

which, as desired, decays with the worldsheet radial distance. We see that in order to satisfy both the conditions $V(r_I) < 0$ and $E < 0$ simultaneously, the restriction on $f^{''}(\sigma_0)$ is

$$\frac{\omega^2}{4} < f''(\sigma_0) < \frac{\omega^2}{2} \tag{5.25}$$

that is the required stability condition for a Schwarzschild brane. Since ω^2 is positive definite, the Schwarzschild brane will be stable if the warp factor has a minimum at the brane location, with the value of $f''(\sigma_0)$ lying within a fixed range given by the above equation.

A second linearly independent solution for Eq (5.22) can be obtained by taking note on the fact that the first solution behaves like $F(r_I) = R_{10}(r_I) \sim e^{-2kr_I}$ [25]. With the help of the Wronskian [26], the second solution is found to be

$$F_2(r_I) \sim \frac{e^{2kr_I}}{4k} \tag{5.26}$$

which is clearly diverging and, as a result, is ruled out by the stability criteria. Hence, the only acceptable solution for Φ is provided by Eq (5.24).

As for the cosmological case, we have, in the above discussion, obtained the Jacobi equations for generic, static, spherically symmetric branes, which can always be used to understand extremality and stability in cases other than Schwarzschild.

B. Asymmetrically warped spacetimes

As mentioned before, the asymmetrically warped spacetimes [27, 28] provide a generalisation of the usual warped extra dimensions. The generalisation is realted to having two different warp factors, one associated with time and another with the 3-space, both functions of the extra dimension alone. Such a generalisation does have serious problems, *e.g,* an apparent Lorentz violation of the 4-dimensional equations [27]. Despite these problems, it is interesting to study different aspects of these models from the geometrical point of view [28]. Here we intend to study the issue of extremality and stability for such asymmetrically warped spacetimes.

Considering the background metric to be of the form

$$dS_5^2 = -e^{2f(\sigma)}dt^2 + e^{2g(\sigma)}(dx^2 + dy^2 + dz^2) + d\sigma^2 \tag{5.27}$$

and the induced metric on the brane as

$$dS_I^2 = -e^{2f(\sigma_0)}dt^2 + e^{2g(\sigma_0)}(dx^2 + dy^2 + dz^2) \tag{5.28}$$

where $f(\sigma)$ and $g(\sigma)$ are the two warp factors, the non-zero components of the normalised tangent vectors read

$$E^t_\tau = e^{-f(\sigma)} \quad ; \quad E^x_{x_1} = E^y_{y_1} = E^z_{z_1} = e^{-g(\sigma)} \tag{5.29}$$

whereas the normal vector is chosen as the same as before. Subsequently the non-zero components of the extrinsic curvature tensor turn out to be

$$K_{\tau\tau} = -f'(\sigma) \quad ; \quad K_{x_1x_1} = K_{y_1y_1} = K_{z_1z_1} = g'(\sigma) \tag{5.30}$$

and the relevant Ricci tensor component is the $\sigma\sigma$ part

$$R_{\sigma\sigma} = -\left[(f^{''}(\sigma) + f^{'2}(\sigma)) + 3(g^{''}(\sigma) + g^{'2}(\sigma))\right] \tag{5.31}$$

With these the 1st and 2nd variation equations lead to the following conditions for asymmetric warping :

$$f^{'}(\sigma_0) + 3g^{'}(\sigma_0) = 0 \tag{5.32}$$

$$f^{''}(\sigma_0) + 3g^{''}(\sigma_0) > 0 \tag{5.33}$$

In this scenario we find no separate minima for the two different warp factors but the stability requirement imposes extra constraints on them. As for example, for a constraint relation of the form $f(\sigma) = \nu g(\sigma)$ [28] equations (5.32) and (5.33) look a bit more familiar :

$$(1 + 3/\nu)f^{'}(\sigma_0) = 0 \tag{5.34}$$

$$(1 + 3/\nu)f^{''}(\sigma_0) > 0 \tag{5.35}$$

The above equations reveal that the brane can be extremal if either of the two conditions $\nu = -3$, $f^{'}(\sigma_0) = 0$ is satisfied. But the first condition, namely, $\nu = -3$, leads to a trivial solution where both $f(\sigma)$ and $g(\sigma)$ turn out to be constant. Hence for a stable extremal brane with linear dependence between warp factors, the stability criteria demands that $f(\sigma)$ (and consequently, $g(\sigma)$) should have a minimum on the brane location. With the solutions from [28]

$$f(\sigma) = \frac{\nu}{\eta}\ln(\eta\sigma + C) \quad ; \quad g(\sigma) = \frac{1}{\eta}\ln(\eta\sigma + C) \tag{5.36}$$

It can be verified that $f(\sigma)$ or $g(\sigma)$ has no minimum anywhere. Hence, this type of asymmetrically warped vacuum solutions are not stable.

C. Branes with co-dimension 2

Brane models with co-dimension 2 have several advantages : all the standard model fields and gravity can be localised on the same brane [29] and the negative tension brane (in the original RS models) is no longer required to solve the so-called 'Hierarchy problem' [30]. In this setup we have two extra dimensions and two σ-dependent warp factors, one associated with the flat 4D part and another with the sixth dimension θ. The background metric, now 6-dimensional, is taken as [29, 30, 31]

$$dS_6^2 = e^{2f(\sigma)}(-dt^2 + dx^2 + dy^2 + dz^2) + d\sigma^2 + e^{2g(\sigma)}d\theta^2 \tag{5.37}$$

and the induced metric representing the 4-dimensional flat brane as

$$dS_I^2 = e^{2f(\sigma_0)}(-dt^2 + dx^2 + dy^2 + dz^2) \tag{5.38}$$

Obviously, the tangent vectors are the same as those in Eq (4.1) and, without loss of generality, we make our choice of normal vectors, satisfying the normalisation conditions, as

$$n_1 = (0, 0, 0, 0, 1, 0) \quad ; \quad n_2 = (0, 0, 0, 0, 0, e^{-g(\sigma)}) \tag{5.39}$$

With this choice, K^2_{ab} becomes identically zero and the 1st variation equation (2.7) for K^1_{ab} reveals that

$$f^{'}(\sigma_0) = 0 \tag{5.40}$$

that is to say that an extremal brane has to be located $\sigma = \sigma_0$, the extremum of $f(\sigma)$. This condition is identical to the one derived for co-dimension 1 scenario. It can be checked that for any other choice of the normal vectors, every component of K^2_{ab} is related to the corresponding component of K^1_{ab} by $K^2_{ab} = F(\sigma) \times K^1_{ab}$, the trace of which gives no extra equation as such. Hence our choice of normal vectors are indeed justified and sufficient for

the present discussion. One might have wondered why no constraint is there on the warp factor associated with the 6th dimension θ, even if the metric looks asymmetrically warped in a sense that there are two different warp factors. The fact that the 4-dimensional part that represents the brane is associated with a single warp factor independent of θ is the obvious answer to it.

In order to calculate the effective mass matrix $(M^2)^{ij}$ for two extra dimensions, one requires both Riemann and Ricci tensors (along with the extrinsic curvature). The relevant components of these quantities are listed below.

$$R_{\sigma\theta\sigma\theta} = R_{\theta\sigma\theta\sigma} = -e^{2g(\sigma)}[g^{''}(\sigma) + g^{'2}(\sigma)] \tag{5.41}$$

$$R_{\sigma\sigma} = -4[f^{''}(\sigma) + f^{'2}(\sigma)] - [g^{''}(\sigma) + g^{'2}(\sigma)] \tag{5.42}$$

$$R_{\theta\theta} = -e^{2g(\sigma)}[g^{''}(\sigma) + g^{'2}(\sigma) + 4f^{'}(\sigma)g^{'}(\sigma)] \tag{5.43}$$

With these Eq (2.15) now reduces to two equations for two scalars Φ^1 and Φ^2 :

$$\left[\Box - 4f^{''}(\sigma_0)\right]\Phi^1 = 0 \tag{5.44}$$

$$\left[\Box - 4f^{'}(\sigma_0)g^{'}(\sigma_0)\right]\Phi^2 = 0 \tag{5.45}$$

The first equation immediately provides us with the stability condition as before : the extremum of $f(\sigma)$ has to be a minimum at the location of the brane. On the other hand, $f^{'}(\sigma_0)$ being zero, the second equation reveals that the perturbations about θ are always stable. Once again the obvious reason behind it is that the warp factors are functionally independent of θ.

To study an example, let us analyze the model derived in the first reference of [31] where

$$f(\sigma) = \cosh^{4/5}(k\sigma) \quad ; \quad g(\sigma) = g_0\frac{\sinh^2(k\sigma)}{\cosh^{6/5}(k\sigma)} \tag{5.46}$$

It is found that $f(\sigma)$ has a minimum at $\sigma = 0$. As a consequence, this braneworld model with co-dimension 2 turns out to be both extremal and geometrically stable.

VI. SUMMARY AND OUTLOOK

In this article, we have obtained the criteria under which an embedded brane (in a warped background) can be extremal and stable against small normal perturbations. We have applied these criteria to several different flat braneworld models in warped five dimensional backgrounds and found that quite a few of the well–known models do not seem to meet them.

Furthermore, we have generalised our formalism to other related contexts such as symmetrically warped branes with curvature (cosmological branes and static, spherically symmetric branes), asymmetrically warped braneworlds and models with codimension 2. For each of the above cases we have obtained the extremality and stability criteria and illustrated them with some examples.

It must be mentioned that there can be different ways to embed a lower dimensional manifold in a given higher dimensional background. We have chosen the simplest (and widely used) embedding throughout– more complicated embeddings giving rise to nontrivial and different induced metrics can yield quantitatively different results. Additionally, it is also important to state that it is not always necessary for the action to be Nambu–Goto (which is, again the simplest and widely used choice). There can be additional terms such as rigidity corrections, terms involving extrinsic curvature etc. which might change the extremality and stability criteria. Finally, an issue which has not been dealt here is that of large deformations (eg. formation of cusps and kinks on the worldsheet) which requires the use of the so–called generalised Raychaudhuri equations[32]. We hope to address these issues in future investigations.

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